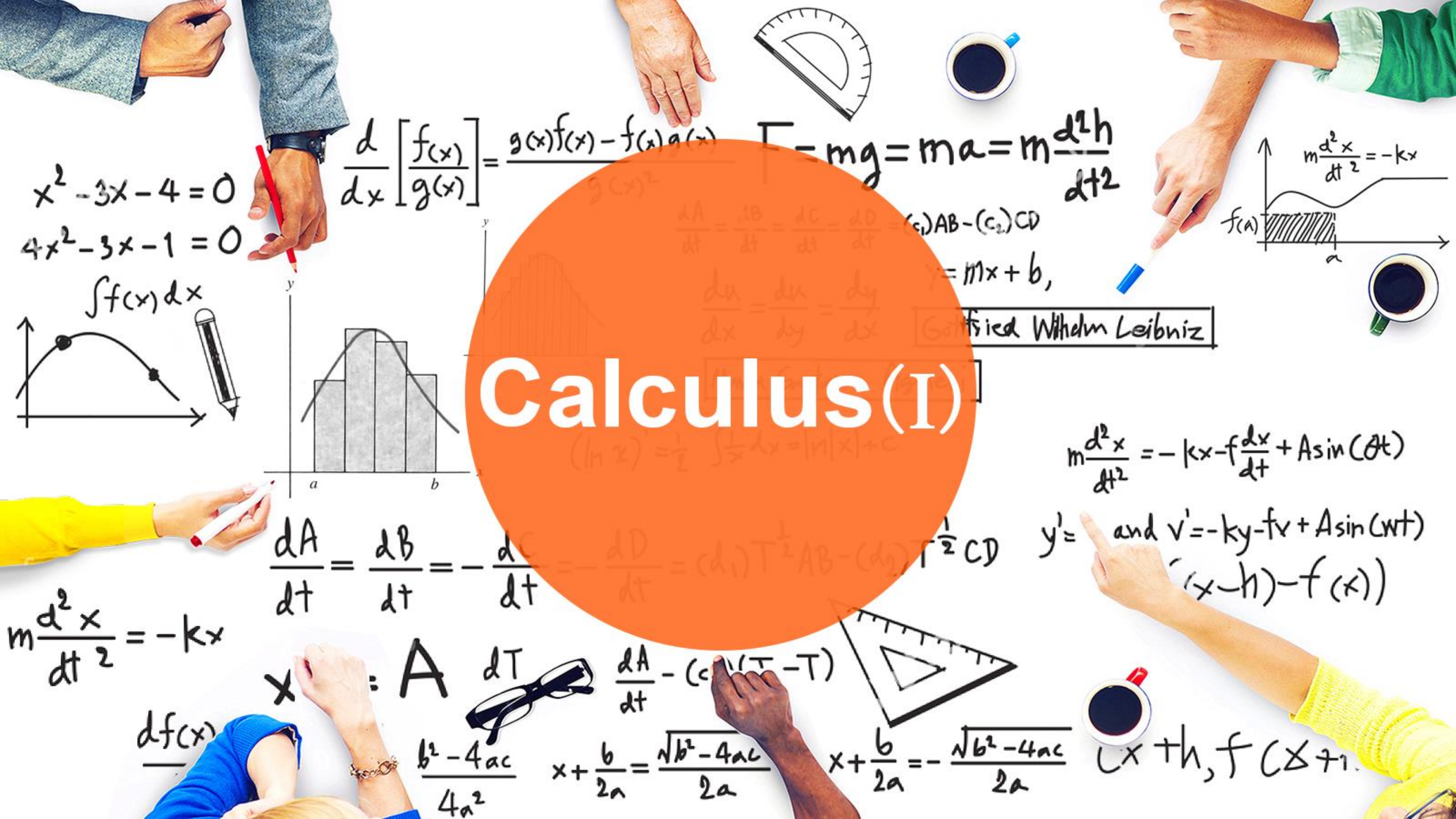
Calculus(Ⅰ)



x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x) dx
F = mg = ma = m d²h/dt²
y = mx + b,
Gottfried Wilhelm Leibniz
m d²x/dt² = -kx
m d²x/dt² = -kx - f dx/dt + Asin(ωt)
and v' = -ky - fv + Asin(ωt)
dA/dt = dB/dt = - dC/dt
x + b/2a = √b² - 4ac / 2a
x + b/2a = - √b² - 4ac / 2a

Continuity of Functions

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The discontinuous point and kinds

Def :

(1) $\lim_{x \to c} f(x)$ exists,

(2) $f(c)$ exists, (c is in the domain of f)

(3) $\lim_{x \to c} f(x) = f(c)$

Any one of these three fails, then f is discontinuous at c.

Two kinds of discontinuity:

I discontinuous point (discontinuity point of the first kind):

$f(x_0-0)$ and $f(x_0+0)$ both exist,

If $f(x_0-0) = f(x_0+0)$, x_0 removable discontinuity.

If $f(x_0-0) \neq f(x_0+0)$, x_0 jump discontinuity.

II discontinuous point (discontinuity point of the second kind):

$f(x_0-0)$ and $f(x_0+0)$ at least one does not exist.

If one of the two limits is ∞, x_0 infinity discontinuity.

If one of two limits is changeable, x_0 oscillatory discontinuity.

infinity discontinuity

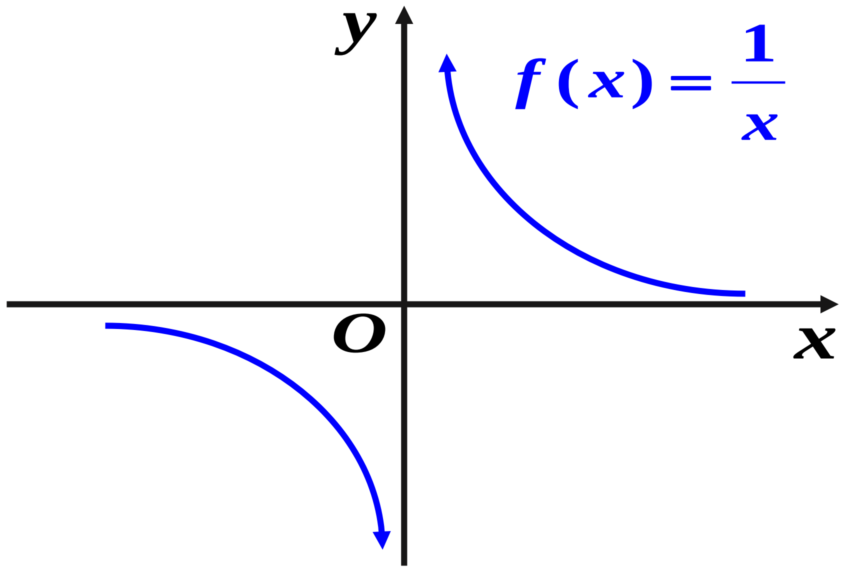
Eg 1: Function $f(x)=\dfrac{1}{x}$,

$f(c)$ does not exist, (c is in the domain of f)

$f(x)$ at $x=0$ is not defined,

$f(0)$ does not exist $\Rightarrow$ discontinuity.

and $\lim\limits_{x\to 0^+} f(x)=+\infty$, $\lim\limits_{x\to 0^-} f(x)=-\infty$

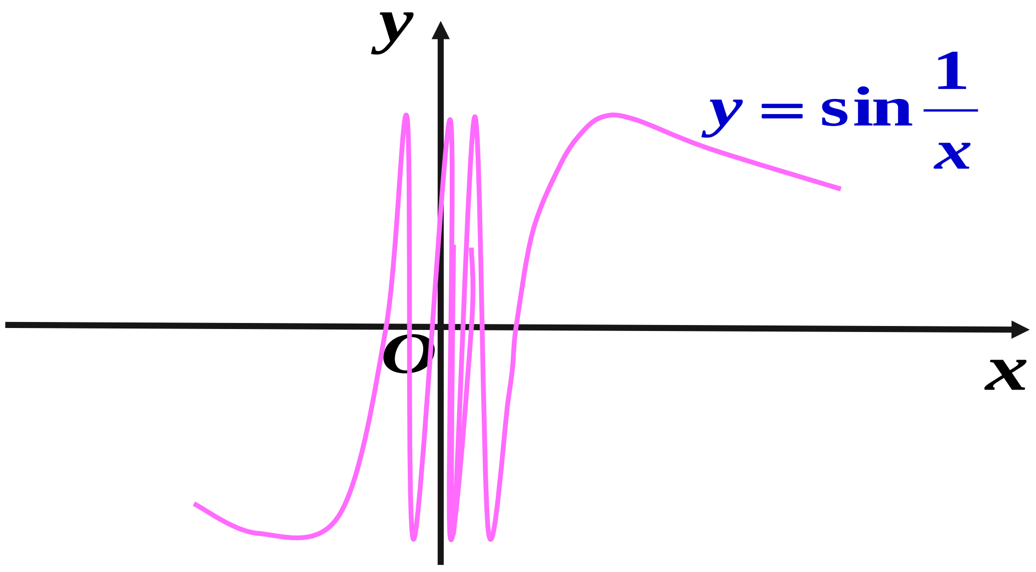


$x=x_0$ infinity discontinuity.

oscillatory discontinuity

Eg 2: $\lim_{x \to c} f(x)$ does not exist

Function $f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$



(1) $f(0)$ exists;

(2) $\lim_{x \to 0} \sin \dfrac{1}{x}$ does not exist;

$x \to 0, \sin \dfrac{1}{x}$ is changeable between 1 and -1.

$x = 0$ oscillatory discontinuity.

jump discontinuity

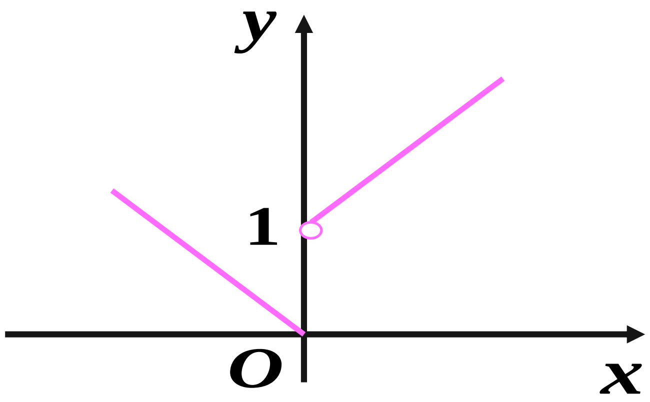
Eg 3: Function $f(x)=\begin{cases}-x, & x\le 0,\\ 1+x, & x>0,\end{cases}$

$\lim_{x\to c} f(x)$ does not exist

(1) $f(0)$ exists;

(2) $\lim_{x\to 0^-}(-x)=0 \quad \lim_{x\to 0^+}(1+x)=1$

$f(0-0)\neq f(0+0)$, limdoes not exist.



$x=0$ jump discontinuity.

removable discontinuity

Eg 4: Discuss function $\lim_{x \to x_0} f(x) \neq f(x_0)$,

$$f(x) = \begin{cases} 2\sqrt{x}, & 0 \le x < 1, \\ 1, & x = 1 \\ 1 + x, & x > 1, \end{cases}$$ is continuous at $x = 1$ or not.

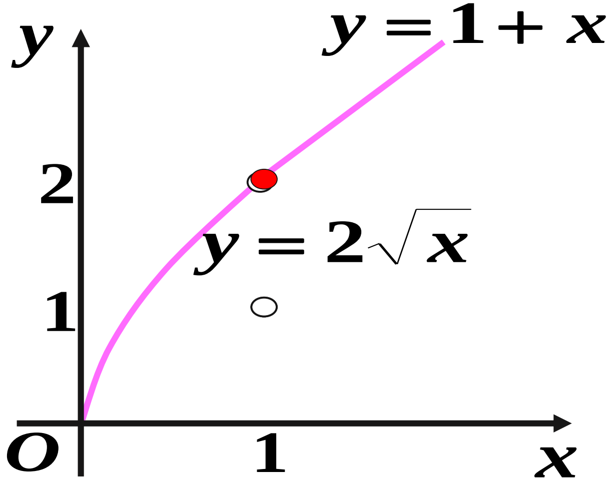
$\because f(1) = 1$

$f(1-0) = 2, \quad f(1+0) = 2,$

$\therefore \lim_{x \to 1} f(x) = 2$ $= f(1) = 2$

$\therefore x = 1$ **discontinuity.**

and is removable discontinuity.



Then, $f(x) = \begin{cases} 2\sqrt{x}, & 0 \le x < 1, \\ 1 + x, & x \ge 1, \end{cases}$ **is continuous at** $x = 1$.

Note

As to removable point x_0, let $\lim_{x \to x_0} f(x) = A$,

And supplement or change the function value of x_0

Such that equal to A, x_0 is continuous point.

removable discontinuity

Eg : $y = \frac{x^2 - 1}{x - 1}$ is not defined at $x = 1$,

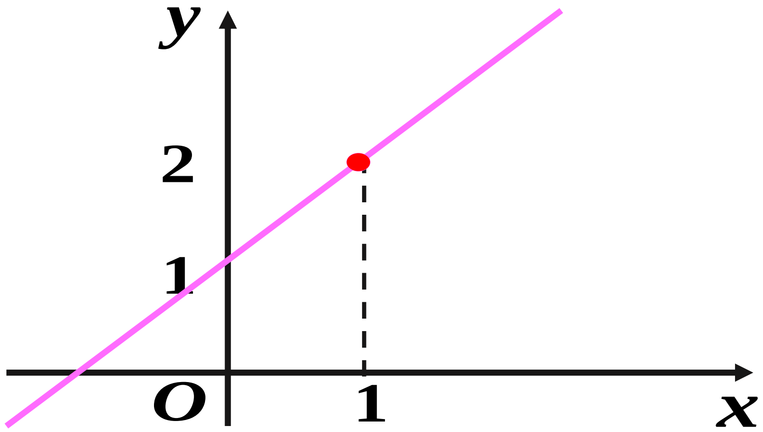
$x = 1$ discontinuity.

But $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2$

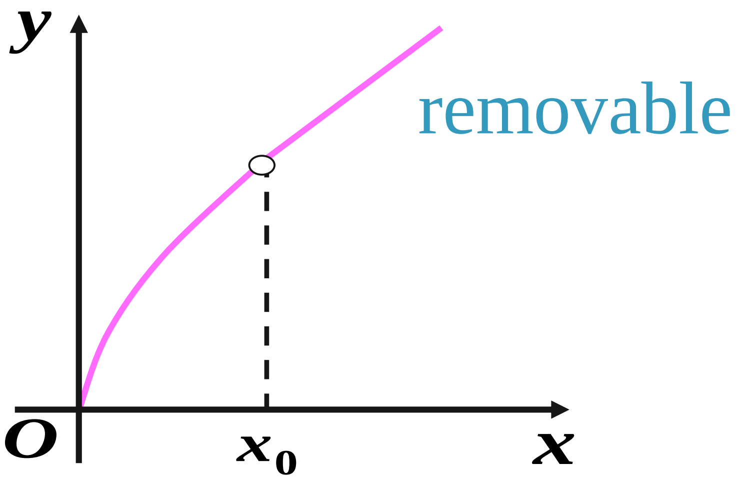
If supplement: let $f(1) = 2$,

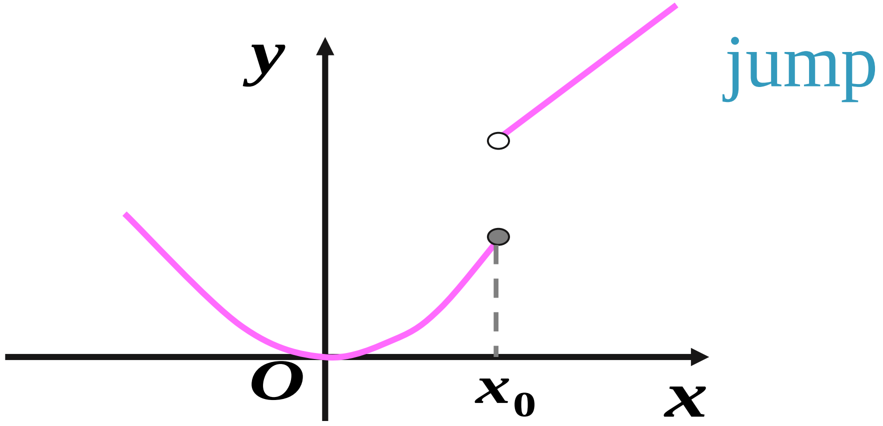
Then function is continuous at $x = 1$.

So $x = 1$ is called removable.

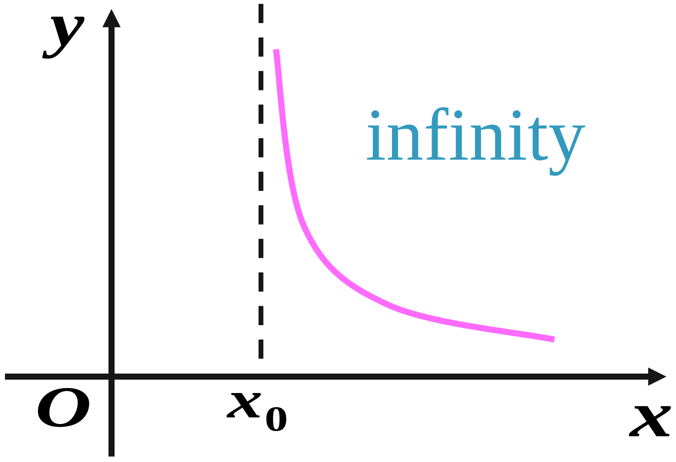


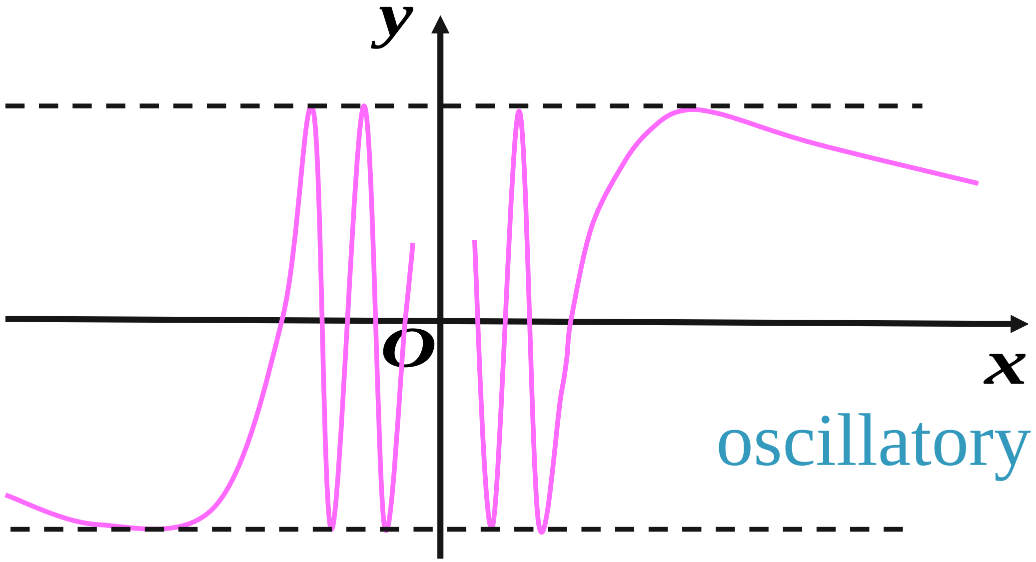
The first kind





The second kind





Relationship of continuity and limit:

$f(x)$ is continuous at x_0 $\Longrightarrow$ (and $\not\Longleftarrow$) the limit of $f(x)$ exist at x_0

Exercise

Say something about kind.

Find the discontinuity of $f(x)=\dfrac{1}{1-e^{\frac{x}{1-x}}}$,

When $x=0, x=1,$ not defined, discontinuity.

$x=0,$ by $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{1}{1-e^{\frac{x}{1-x}}}=\infty,$

So $x=0$ infinity discontinuity.

$x=1,$ by $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\dfrac{1}{1-e^{\frac{x}{1-x}}}=0$ $\quad \frac{x}{1-x}\to +\infty$

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}\dfrac{1}{1-e^{\frac{x}{1-x}}}=1$ $\quad \frac{x}{1-x}\to -\infty$

So $x=1$ jump discontinuity.

Exercise

Let $f(x)=\begin{cases} \dfrac{\sin x}{x} & x<0 \\ a & x=0 \\ b+x\sin\dfrac{1}{x} & x>0 \end{cases}$ $a,b=?$

(1) $\lim\limits_{x\to 0} f(x)$ exist; (2) $f(x)$ is continuous at $x=0$.

By $\lim\limits_{x\to 0^-} f(x)=1$, $\lim\limits_{x\to 0^+} f(x)=b$, so

(1) $\lim\limits_{x\to 0} f(x)$ exist; only need to

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)$, namely, $b=1$ (a any number).

(2) $f(x)$ is continuous at $x=0$. Only need to

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)$, namely $a=b=1$.

Intermediate Value Theorem

Lemma (Existence Theorem of Equation Root)

Zero Point Theorem

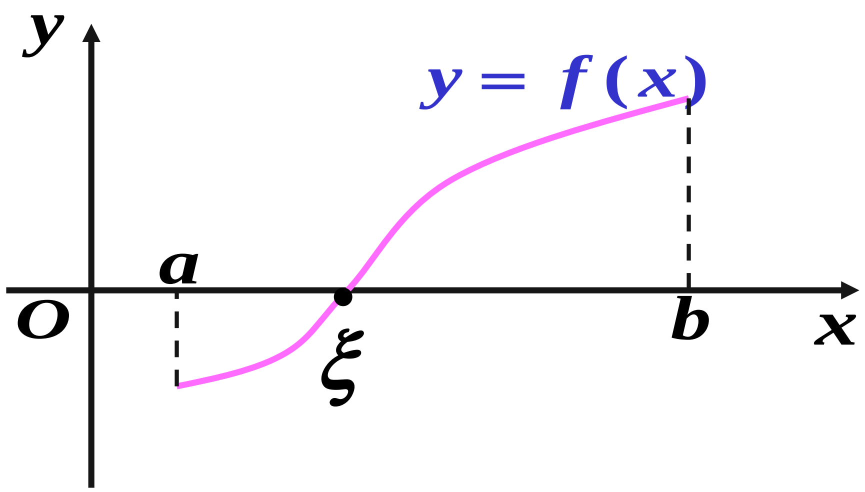
Let $f(x) \in C[a,b]$,

$f(a) \cdot f(b) < 0$ ($f(a)$ and $f(b)$ have different signs)

then there is at least one point $\xi \in (a,b)$, $f(\xi) = 0$,

ξ is the root of equation $f(x) = 0$, sometimes called zero point

Geometrical meaning:



Theorems of Continuity of Functions

Th F

$$\left.\begin{array}{l}\text{If } f \text{ is defined on } [a,b] \\ f \text{ is continuous on } [a,b] \\ \forall w \in [f(a), f(b)]\end{array}\right\} \Rightarrow \exists c \in [a,b], \text{such that } f(c) = w.$$

Proof : (1) $f(a) = f(b) \Rightarrow c = a \Rightarrow f(c) = w, c \in [a,b].$

Theorems of Continuity of Functions

Th F

$$\left.\begin{array}{l}\text{If } f \text{ is defined on } [a,b] \\ f \text{ is continuous on } [a,b] \\ \forall w \in [f(a), f(b)]\end{array}\right\} \Rightarrow \exists c \in [a,b], \text{such that } f(c) = w.$$

Proof : (2) $f(a) \neq f(b)$

Auxiliary function $\phi(x) = f(x) - w,$ then $\phi(x) \in C[a,b],$

and $\phi(a) = f(a) - w = f(a) - w,$ Zero Theorem

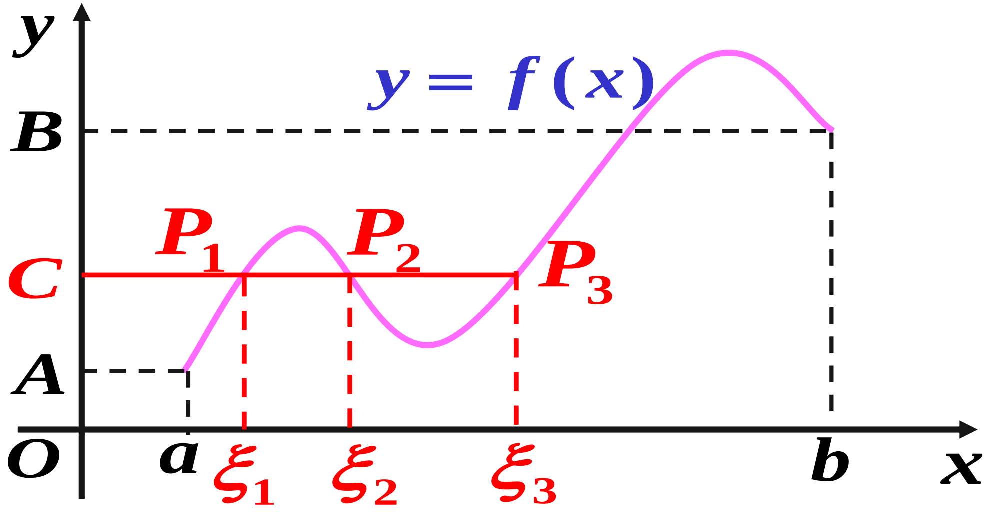
and $\phi(b) = f(b) - w = f(b) - w,$

$\therefore \phi(a) \cdot \phi(b) < 0,$ $\exists\, c \in (a,b),$ such that

$\phi(c) = 0,$ namely, $\phi(c) = f(c) - w = 0,$ $\therefore f(c) = w.$

Geometrical meaning:

There is at least one intersection point of $y = f(x)$ and line $y = C$



Questions and Answers

Eg 1: Show that the equation $x^3 - 8x + 1 = 0$ has at least one root in $(0,1)$.

Let $f(x) = x^3 - 8x + 1 \in C[0,1]$

and $f(0) = 1 > 0$, $f(1) = -6 < 0$,

By Intermediate Value Theorem,

$\exists\, c \in (0,1)$, such that $f(c) = 0$, namely, $c^3 - 8c + 1 = 0$,

$\therefore\ x^3 - 8x + 1 = 0$ has at least one root in $(0,1)$.

Questions and Answers

Eg 2: $f(x) \in C[a,b]$, and $f(a) < a$, $f(b) > b$.

Prove $\exists \xi \in (a,b)$, such that $f(\xi) = \xi$.

Auxiliary function

Let $F(x) = f(x) - x$, $F(x) \in C[a,b]$,

and $F(a) = f(a) - a < \mathbf{0}$,

$F(b) = f(b) - b > \mathbf{0}$,

$\exists\, \xi \in (a,b)$, s.t. $F(\xi) = f(\xi) - \xi = \mathbf{0}$,

Namely, $f(\xi) = \xi$.

Questions and Answers

Eg 3: $f(x) \in C[a,b], \alpha, \beta > 0,$

prove: $\exists \xi \in [a,b]$, sucht that $f(\xi) = \dfrac{\alpha f(a) + \beta f(b)}{\alpha + \beta}.$

(1) $f(a) = f(b), \quad \xi = a.$

(2) $f(a) < f(b), \quad (f(a) > f(b),$ similar way$).$

Let $\mu = \dfrac{\alpha f(a) + \beta f(b)}{\alpha + \beta}$

Obviously, $f(a) < \mu < f(b),$

Intermediate Value TH

$\exists \xi \in (a,b)$, s.t. $f(\xi) = \mu,$

Namely, $f(\xi) = \dfrac{\alpha f(a) + \beta f(b)}{\alpha + \beta}.$

Questions and Answers

Eg 4: $f(x), g(x) \in C[a,b]$, and $f(a) < g(a), \lim\limits_{\delta x \to 0} f(b) > g(b)$,

prove: $\exists \xi \in [a,b], \quad s.t. \quad f(\xi) = g(\xi).$

Let $F(x) = f(x) - g(x)$, then

$F(x) \in C[a,b]$, and

$F(a) = f(a) - g(a) < \mathbf{0};$

$F(b) = f(b) - g(b) > \mathbf{0}.$ Zero Theorem

$\exists \xi \in (a,b), \quad s.t. F(\xi) = 0,$ namely, $f(\xi) = g(\xi).$

Continuity of Functions